

# Abrikosov vortex lattice nonlinear dynamics and its stability in the presence of weak defects.

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Abrikosov vortex lattice dynamics in a superconductor with weak defects is studied taking into account gyroscopic (Hall) properties. It is demonstrated that interaction of the moving lattice with weak defects results in appearance of the additional drag force,  $F$ , which is  $F(V) \propto \sqrt{V}$  at small velocities, while at high velocities  $F(V) \propto 1/\sqrt{V}$ . Thus, the total drag force can be a nonmonotonic function of the lattice translational velocity. It leads to the velocity dependence of the Hall angle and a nonlinear current-voltage characteristic of the superconductor. Due to the Hall component in the lattice motion, the value  $[dF(V)/dV]^{-1}$  does not coincide with the differential mobility  $dV/dF_e$ , where  $F_e$  is the external force acting on the lattice. The estimates suggest that condition  $dV/dF_e < 0$  can be met with difficulty in contrast to  $dV/dF < 0$ . The instability of the lattice motion regarding nonuniform perturbations has been found when the value of  $(-dF/dV)$  is greater than a certain combination of elastic moduli of the vortex lattice, while the more strict requirement  $dV/dF_e < 0$  is still not fulfilled.

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## I. INTRODUCTION

Vortex dynamics in superconductors is still a controversial matter in physics of superconductivity. As a rule dynamics of Abrikosov vortices was considered in the overdamped regime,<sup>1</sup> when the viscosity is strong and it namely determines lattice dynamics. The Hall force was generally neglected in conventional superconductors, where viscosity is high. Interest in the Hall force has quickened in the past few years with fabrication of high- $T_c$  superclean single crystals with the large Hall angle.<sup>2</sup> Moreover, among other things, Hall properties are manifested in special collective modes existing in the vortex system. These modes were observed in resonant experiments with  $^4\text{He}$ .<sup>3</sup> The interest to these modes as applied to high- $T_c$  superconductors has increased considerably in recent years in connection with experiments in BSCCO<sup>4</sup> and YBCO<sup>5</sup> compounds in which resonant effects were observed. In YBCO system resonances occur most likely because of cyclotron vortex modes, while the resonant effects in BSCCO are associated either with Josephson plasma modes emerging there owing to Josephson coupling of adjacent  $\text{CuO}_2$  layers<sup>6</sup> or with the vortex modes excitations.<sup>7</sup>

The vortex modes may manifest themselves not only in resonances but also in dissipation of the moving vortex lattice caused by excitation of these modes. This effect is owing to irreversible transfer of the translational vortex motion energy into elementary bosonic excitations - the quanta of vortex modes in the presence of the vortex-defects interaction. For the single vortex it has been demonstrated that excitation of vortex modes results in an anomalous behavior of the friction force,  $F_v$ , which is divergent at  $V \rightarrow 0$ .<sup>8</sup> The next step is study of vortex

modes excited in the moving vortex lattice.

In the present work we investigate vortex lattice dynamics in the presence of weak defects. We are interested in an additional effect of dissipation induced by the vortex lattice motion and interaction with weak defects, which are not strong enough to pin the vortex lattice, but this motion results in excitation of normal vibrational modes and irreversible transfer of the interaction energy to these modes (Landau damping).

Due to the additional dissipation effect nonlinearity of the lattice motion has been revealed and it has been demonstrated that the nonlinear regime may be even dominant one in lattice dynamics being compared with the bare linear dissipation. When the lattice is the subject of the Hall force we predict appearance of a lattice motion instability. The Hall angle is found to be lattice velocity dependent and the higher velocity the larger Hall angle as far as it is saturated.

The paper is composed of five sections as follows. In Sec.II we describe a microscopical model of vortex lattice dynamics with the Hall and viscous contributions. In Sec. III we study the steady motion of the vortex lattice and discuss the stability of this motion. Discussion of the results of the model is presented in Sec.IV. Finally, the summary is given in Sec. V.

## II. MODEL

Let us consider dynamics of the vortex lattice (formed by Abrikosov vortices parallel to the  $z$ -axis). Let vortices in equilibrium to be placed in the points of an ideal lattice,  $\vec{l}$ . Dynamics of the vortex lattice can be described on the basis of an effective equation for the 2D vector

$\vec{u} = \vec{u}_l(z, t)$ , lying in the  $xy$ -plane and describing the displacement of the  $\vec{l}$ -th vortex in the lattice (see Fig.1).<sup>1</sup> The following Lagrangian determines nondissipative dynamics of  $\vec{u}$ :

$$L\{\vec{u}\} = \sum_l \int dz \left[ \frac{H}{2} \left( u_{l,x} \frac{\partial u_{l,y}}{\partial t} - u_{l,y} \frac{\partial u_{l,x}}{\partial t} \right) \right] - W\{\vec{u}_l\} - U_{imp}\{\vec{u}\}. \quad (1)$$

Here  $H$  is the Hall constant of the single vortex per unit length,  $W = W\{\vec{u}_l\}$  is the energy of the ideal (defectless) deformed lattice,  $U_{imp}\{\vec{u}_l\}$  defines the energy of interaction of the vortex with defects (its structure will be considered below). To describe dynamics in the linear approximation  $W\{\vec{u}_l\}$  should have the following structure:

$$W\{\vec{u}_l\} = \frac{1}{2} \sum_{\vec{l}} \int \left\{ \kappa \left[ \frac{\partial \vec{u}_l(z, t)}{\partial z} \right]^2 + \sum_{\vec{l}' \neq \vec{l}} U^{ik}(\vec{l} - \vec{l}') \delta u_{l,\nu}^i \cdot \delta u_{l',\nu}^k \right\} dz, \quad (2)$$

where  $\vec{\delta u}_{l,\nu} = \vec{u}_l(z, t) - \vec{u}_{l'}(z, t)$ ,  $U^{ik}(\vec{l})$  are components of the force tensor which are expressed through the second derivatives of a well-known potential of the vortex-vortex interaction with respect to equilibrium vortex positions,<sup>9</sup>  $\kappa$  is the energy per unit length. We neglect in the above expression of the Lagrangian the inertial term, which comprises the vortex mass because, as was demonstrated in<sup>8</sup>, the vortex mass plays a substantial role in the dissipation of the moving vortex lattice only at high enough velocities. In the framework of a macroscopic approach, i.e. if  $\vec{u}_l$  varies insignificantly over a distance of the order of the lattice constant  $a_v$ , the last expression (2) goes over to the standard expression as with the theory of elasticity,

$$W\{\vec{u}\} = \int dx dy dz \left[ \frac{c_{11}}{2} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right)^2 + \frac{c_{66}}{2} \left( \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right)^2 + \frac{c_{44}}{2} \left( \frac{\partial \vec{u}}{\partial z} \right)^2 \right], \quad (3)$$

where  $c_{11}, c_{44}, c_{66}$  are the elastic moduli of the lattice,  $c_{44}$  can be expressed through  $\kappa$ ,  $c_{44} = \kappa/a_v^2$ .

Each vortex moving in the medium experiences the action of the drag force and the rate of its energy dissipation is proportional to  $(\partial \vec{u}_l / \partial t)^2$ . For this reason, we choose the dissipative function in the form:

$$Q = \frac{\beta}{2} \sum_l \int \left( \frac{\partial \vec{u}_l}{\partial t} \right)^2 dz, \quad (4)$$

where  $\beta$  is the dissipation coefficient per unit length of the vortex. When the motion of the vortex lattice is

translational and steady ( $\vec{u}_l(z, t) = u_l(0) + \vec{V}t$ , where  $\vec{V}$  is the translational vortex lattice velocity) the dissipative function yields the friction force per unit length of the vortex,  $\vec{F} = -\vec{V}(Q/V^2) = -\vec{V}\beta$ .

In the case of "weak" defects the microscopical origin of defects is not essential.<sup>8</sup> We determine interaction between the vortex lattice and defects in the crystal as a variation of a local critical temperature,  $T_c$ , depending on coordinates. This can be taken into account by introducing a coordinate-dependent coefficient  $[a_0 + F(\vec{r})]$  into the Ginzburg-Landau expansion.<sup>1</sup> We assume that the distribution of the order parameter in the vortex does not change during the vortex motion and is described by a known function  $f(r_\perp)$ , where  $r_\perp^2 = x^2 + y^2$ .<sup>9</sup> Thus, regarding the displacement of the  $l$ -th vortex, we can write the vortex lattice energy  $\sum_l |\Psi_0|^2 f(r_{l\perp})$ , where  $r_{l\perp} = [(x - l_x - u_{lx})^2 + (y - l_y - u_{ly})^2]^{1/2}$  and  $\Psi_0$  is the order parameter. In this case, the energy  $U_{imp}$  associated with crystal inhomogeneities can be written in the terms of the function  $f(r_\perp)$  in the form of a functional of the vortex lattice displacement:

$$U_{imp} = \sum_{\vec{l}} \int d\tilde{x} d\tilde{y} dz f(r_\perp) F[\tilde{x} + l_x + u_{lx}, \tilde{y} + l_y + u_{ly}, z], \quad (5)$$

where  $\tilde{x} = x - l_x - u_{lx}$ ,  $\tilde{y} = y - l_y - u_{ly}$ .

Finally, one arrives at the equation for the displacement of the  $l$ -th vortex,  $\vec{u}_l(z)$ . It is presented in the form:

$$H \left( \vec{z} \times \frac{\partial \vec{u}_l}{\partial t} \right) = - \frac{\delta W\{\vec{u}_l\}}{\delta \vec{u}_l} - \beta \frac{\partial \vec{u}_l}{\partial t} + \vec{F}_{l,imp}. \quad (6)$$

Here the left hand side is the dynamical term governed by the Hall (gyroscopic) constant, the terms on the right hand side are forces acting on the  $l$ -th vortex because due to interaction with other vortices in the ideal lattice, friction and the presence of defects.

Using the Eq. (5), the force caused by defects can be written in the form of the Fourier expansion in  $x$  and  $y$ :

$$\vec{F}_{l,imp} = \frac{1}{\Omega} \sum_{\vec{q}} i \vec{q}_\perp f(q_\perp) \mathcal{F}(\vec{q}) e^{i \vec{q}_\perp \vec{l} + i q_z z} e^{i \vec{q}_\perp \vec{u}_l(z)}, \quad (7)$$

where  $\Omega = L_x L_y L_z$  is the superconductor volume,  $\mathcal{F}(\vec{q})$  is the Fourier transform of the function  $F(\vec{r})$ , which defines inhomogeneities in the system, see Eq.(5),  $\vec{q}_\perp = (q_x, q_y, 0)$ ,  $f(q_\perp)$  is the vortex form-factor.

In the absence of dissipation and defects the equation (6) can be simplified and because of translational invariance the force tensor components  $U^{ik}$  depend on  $\vec{l} - \vec{l}'$  only and do not depend on  $z$ . Then, the equation of motion can be presented as it is done in the lattice dynamics theory, i.e. by means of the Fourier transform of the force tensor components.<sup>10</sup> By using the Bloch

theorem it is possible to present the equation of motion in the normal coordinates  $\vec{u}_{q,x(y)}$  considering that  $\vec{u}_{l,x(y)} = \sum_{\vec{q}_\perp, q_z} \vec{u}_{\vec{q},x(y)} \cdot \exp(i\vec{q}_\perp \vec{l} + iq_z z)$ .

The equation of motion acquires the following form:

$$\begin{aligned} H\dot{u}_{q,x} &= C_{yy}(\vec{q}) \cdot u_{q,y} + C_{xy}(\vec{q}) \cdot u_{q,x}; \\ -H\dot{u}_{q,y} &= C_{xx}(\vec{q}) \cdot u_{q,x} + C_{xy}(\vec{q}) \cdot u_{q,y}, \end{aligned} \quad (8)$$

where  $C_{ik} = \sum_l U^{ik}(\vec{l}) \cdot \exp(i\vec{q}\vec{l})$  is Fourier representation of the force tensor  $U^{ik}$ , see (2). Here and below dot denote the time derivative. The Fourier transform of  $\vec{u}_l$  is discrete in the  $xy$ -plane and continuous along the  $z$ -axis, the quantity  $\vec{q}_\perp$  has the sense of a quasimomentum, and  $q_z$  is a regular linear momentum, (we put  $\hbar = 1$ ). The quasimomentum  $\vec{q}_\perp$  there is within the first Brillouin zone,  $|\vec{q}| \leq 1/a_v$ , whereas  $q_z$  is only restricted by the physical reason, i.e.  $q_z \leq 1/\xi$ , where  $\xi$  is the coherence length of the superconductor. Though these values  $\vec{q}_\perp$  and  $q_z$ , have somewhat different physical meanings, below we will use a macroscopical approach and retain these symbols.

The variables  $\vec{u}_{\vec{q}}$  comprise the time dependent multiplier,  $\vec{u}_{\vec{q}} \propto \exp[\pm i\omega(\vec{q})t]$ , where  $\omega(\vec{q})$  is periodic with regard to  $\vec{q}_\perp$ ,  $\omega(\vec{q} + \vec{g}) = \omega(\vec{q})$ ,  $\vec{g}$  is the vector of the reciprocal lattice. Then, one can derive the dispersion law using new parameters  $C_1 = C_{xx} - \Delta$ ,  $C_2 = C_{yy} + \Delta$ ,  $2\Delta = (C_{xx} - C_{yy}) + \sqrt{(C_{xx} - C_{yy})^2 + 4C_{xy}^2}$ :

$$\omega^2 = \frac{1}{H^2} C_1 C_2. \quad (9)$$

At large  $|\vec{q}| \sim 1/a_v$  the dispersion law of vortex lattice oscillations becomes periodical in the  $\vec{q}_\perp$  plane, with a period determined by the reciprocal lattice constant and the maximal value  $q_{x,y} = 1/\xi$  since the coefficients  $C_{xx(y)}$  and  $C_{xy}$  are periodical functions of  $\vec{q}_\perp$ . Near the center of the first Brillouin zone,  $|\vec{q}| \ll 1/a_v$ , the dispersion law (9) can be presented via elastic moduli,  $C_1 = a_v^2 (c_{44}q_z^2 + c_{66}q_\perp^2)$ ,  $C_2 = a_v^2 (c_{44}q_z^2 + c_{11}q_\perp^2)$ :

$$\omega = \frac{a_v^2}{H} \sqrt{(c_{44}q_z^2 + c_{66}q_\perp^2)(c_{44}q_z^2 + c_{11}q_\perp^2)}, \quad (10)$$

which has been also obtained in Refs.8,11. This law corresponds to a low frequency gapless mode in the limit of small  $q_z$  and  $q_\perp$ .

From here on we will employ a Debye-like model, with approximation of the dispersion law by the long wave equation (10) and with taking into account periodicity in the  $q_x, q_y$  directions. The dispersion law for the vortex lattice is schematically drawn in Fig.2. The Debye frequency for  $\omega(q_\perp)$ ,  $\omega_D$ , is marked in Fig.2 by a dotted line.

We have concerned lattice oscillations without dissipation and defects and found the dispersion law of normal modes. The dissipation consideration does not make

the problem more complicated as equations are still linear. However, the presence of defects results in nonlinear equations as the interaction energy (5) and force (7) depend nonlinearly on  $\vec{u}$ . A general solution of this nonlinear equation (6) cannot be found. In line with<sup>8</sup> we can carry out a complete analysis assuming that deviations from the uniform translational and steady motion of vortices due to defects are small. Putting  $\vec{u} = \vec{e}_x Vt + \vec{\tilde{u}}(\vec{r}, t)$  and linearizing (7) in respect to  $\vec{\tilde{u}}$ , we obtain the expression for  $F_{imp}$  without  $\vec{\tilde{u}}_l(z)$ :

$$\vec{F}_{l, imp} = \frac{1}{\Omega} \sum_{\vec{q}} i\vec{q}_\perp f(q_\perp) \mathcal{F}(\vec{q}) e^{i\vec{q}_\perp \vec{l} + iq_z z - iq_x Vt}. \quad (11)$$

Thus, the lattice motion through the nonuniform medium results in excitation of small oscillations of vortices in the lattice (vortex lattice normal modes), which are described by  $\vec{\tilde{u}}$ . It was discussed by the authors of Ref. 12 in the terms of "the lattice melting". Nevertheless, it is essential for us that oscillations bring into appearance of the additional contribution to dissipation of the lattice,  $\Delta Q = (\beta/2) \sum_l \int (\partial \vec{\tilde{u}}_l / \partial t)^2 dz$ , and in turn into the additional drag force,  $\vec{F}(V)$ ,

$$\vec{F}(V) = -\vec{V} \frac{F(V)}{V}, \quad F(V) = \frac{\Delta Q(V)}{V}. \quad (12)$$

The general equation for  $\vec{\tilde{u}}_q$  (compare with (8)) reads

$$\begin{aligned} \tilde{u}_{q,x} &= \frac{1}{D} [(C_{yy} + i\beta q_x V) f_x - (C_{xy} - iH q_x V) f_y]; \\ \tilde{u}_{q,y} &= \frac{1}{D} [(C_{yy} + i\beta q_x V) f_y - (C_{xy} + iH q_x V) f_x], \end{aligned} \quad (13)$$

where  $C_{ik} \equiv C_{ik}(\vec{q})$ ,  $D = (C_1 + i\beta q_x V) \cdot (C_2 + i\beta q_x V) - (H q_x V)^2$ ,  $f_{x,y}$  is related to (11)

$$f_{x,y} = \frac{i}{\Omega} q_{x,y} f(q_\perp) \mathcal{F}(\vec{q}) e^{-iq_x Vt}. \quad (14)$$

It is clearly seen that the condition  $D = 0$  at  $\beta = 0$  defines the dispersion law found in (9), if one would replace  $\omega \rightarrow q_x V$ . It is a typical evidence of resonant character of excitation of normal modes because of the lattice uniform motion.

To analyze specifically a contribution of these small oscillations to dissipation of the moving vortex lattice we assume that inhomogeneity is caused by a system of point defects whose size is smaller than the radius of the vortex core. In this case, the function  $F(x, y, z)$  in (5) can be written as the sum of Dirac delta functions:

$$F(\vec{r}) = \sum_a \alpha \delta(\vec{r} - \vec{r}_a). \quad (15)$$

Here  $\vec{r}_a$  is the coordinate of the  $a$ -th defect and  $\alpha$  characterizes the intensity of interaction of the defect.

The force acting on the  $l$ -th vortex with regard to the model (15) is written as (11) with substitution

$$\mathcal{F}(\vec{q}) = \alpha \sum_a e^{i\vec{q}\vec{r}_a}. \quad (16)$$

By substituting (13) with the taking into account (16) in the dissipative function, employing the transformation

$$\frac{1}{\Omega} \sum_l \int dz \left( \frac{\partial \vec{u}_l}{\partial t} \right)^2 \rightarrow \frac{\Omega}{(2\pi)^3 a_v^2} \int d^3 q (q_x V)^2 \vec{u}_q \vec{u}_q^*, \quad (17)$$

where star denotes the complex conjugation, and carrying out averaging over defects with the help of the relation  $\sum_a \exp(i\vec{q}\vec{r}_a) = N_{imp} \delta_{\vec{q},0}$ , where  $N_{imp}$  is the number of defects,  $\delta_{\vec{q},0}$  is the Kronecker symbol, we can find the additional dissipation of the moving lattice. It may be associated with the drag force  $F(V)$ . In the framework of the model (15) one can obtain the equation for the additional drag force  $F(V)$  per unit length of the vortex

$$F(V) = C_{imp} \frac{\beta V \alpha^2}{(2\pi)^2} \int d^3 q \cdot q_x^2 f^2(q_\perp) \frac{q_\perp^2 [q_x V^2 (\beta^2 + H^2) + C_2^2] + (C_1 + C_2)(q_x \sqrt{\Delta} - q_y \sqrt{C_1 - C_2 - \Delta})^2}{[C_1 C_2 - (\beta^2 + H^2)(q_x V)^2]^2 + \beta^2 (q_x V)^2 (C_1 + C_2)^2}, \quad (18)$$

where  $C_{imp} = N_{imp}/\Omega$  is the defect concentration.

Because of the factor  $\beta V$  appearing in (18) in front of the integral one could assume a linear velocity dependence of the drag force,  $F(V) \propto \beta V$ , where  $\beta$  is some constant. Then, it seems that a consideration of vibrations of vortices associated with interaction with defects results only in a small correction to the *bare* friction force,  $\vec{F}_{bare} = -\beta \vec{V}$ . However, the integral in (18), as well as for the case of the single vortex,<sup>8</sup> contains singularities, therefore, the additional relaxation channel can become significant and even predominant. The most vivid example of this fact is that the additional dissipation can be finite in the limit of zero bare dissipation,<sup>8</sup>  $F(V)$  is finite at  $\beta \rightarrow 0$ , as  $F_{bare} \rightarrow 0$ .

At first glance, this appears as paradoxical. However, such a collisionless damping (Landau damping) stemming from the energy transfer from one mode to another appears in many branches of physics. For instance, flux-flow dynamics caused by excitations of low frequency fermionic modes has been proposed in Ref. 13 and the finite friction force independent of the bare friction is predicted there. The friction force finite at  $V \rightarrow 0$  and caused by excitation of bending oscillation of the domain wall in ferromagnets with microscopic defects was predicted in Ref. 14. As it will be demonstrated below here  $F(V) \propto V^\delta$ ,  $\delta < 1$  at  $V \rightarrow 0$  and  $F(V) > F_{bare}$  at any  $\beta$  and small enough velocities.

This behavior can be explained as following. The expression (18) contains  $\beta$  in the combination  $\beta / \{\beta^2 + G^2 [\vec{q}, (q_x V)^2]\}$ , where the function  $G$  is such that the condition  $G[\vec{q}, (q_x V)^2] = 0$  with the substitution  $q_x V \rightarrow \omega$  defines the frequencies of normal modes of vortex vibrations in the lattice (10). For  $\beta \rightarrow 0$  the expression (18) is transformed into the  $\delta$ -function, and becomes  $\pi \delta \{G[\vec{q}, (q_x V)^2]\}$ . After simple transformations this expression can be reduced to the  $\delta$ -function of the

type  $\delta[(q_x V)^2 - \omega^2(\vec{q})]$ , where  $\omega(\vec{q})$  is the frequency of normal mode (10). Since the equation  $q_x V = \omega(\vec{q})$  possess a solution at any finite velocity the value of  $Q(V)$  can be finite even at  $\beta \rightarrow 0$ .

In the case  $\beta \rightarrow 0$ , it is obvious that the additional dissipation becomes the main source of dissipation in the system. This may be realised if the friction force has a dependence like  $F \propto V^\delta$ ,  $\delta < 1$ , as it was revealed in the case of the single vortex.

There is an alternative explanation of the additional dissipation appearance. The modes excitation in the limit of small dissipation can be described on the basis of momentum and energy conservation. Let us go over to a reference frame moving with vortices. In this reference frame defects move at a velocity  $-\vec{V} = V \vec{e}_x$  parallel to the  $x$ -axis and can transfer the momentum  $\vec{q}$  to the vortex lattice only simultaneously with the energy  $\vec{q}\vec{V} = q_x V$ . This momentum is redistributed between the lattice as a whole and an elementary excitation, and the energy is transferred to elementary excitations only. In particular, in the case of the single vortex the vortex as a whole acquires momentum perpendicular to its axis ( $z$ -axis), while the wave propagating along the vortex obtains the  $z$ -axis momentum projection.

Let us analyse the general expression (18). Note that integration has to be done not only inside the first Brillouin zone. The integration is limited only by cut off  $\tilde{q}_\perp$  over  $1/\xi$ , when the vortex form factor  $f(q_\perp) \neq 0$ . In the case of the vortex lattice, the lattice constant  $a_v$  is usually assumed to be larger than  $\xi$ . In order to resolve the vortex lattice problem one should go beyond the long-wave limit. Generally speaking, one has to integrate (18) taking into account complicated periodical dependencies of  $C_1$ ,  $C_2$ ,  $\Delta$  of the wave vectors, and a non-periodic dependence of  $f(q_\perp)$ , terms with  $q_x V$  and so on. Nevertheless, the situation is simplified in limiting cases, for

example, at small or high velocities.

First we concern the case of small dissipation,  $\beta \ll H$ , for which the conservation law  $\omega(\vec{q}) = q_x V$  defines an area that contributes to the integral in (18). To provide an estimation, we assume that  $c_{11} \sim c_{66} \sim c_\perp$ , then  $\omega H = a_v^2 (c_{44} q_z^2 + c_\perp q_\perp^2)$ , and the condition  $\omega(\vec{q}) = q_x V$  may be written as

$$c_{44} c_\perp q_z^2 + (c_\perp q_x - VH/2a_v)^2 + c_\perp^2 q_y^2 = (VH/2a_v)^2. \quad (19)$$

This expression indicates that at small  $V$  the size of the region in the  $\vec{q}_\perp$  plane, which contributes to the integral in (18) is proportional to  $V$ . If one considers that  $c_{11} \neq c_{66}$ , then of course the above consideration remains true and only the expression (19) becomes more complicated. Thus, at  $V \rightarrow 0$  the region near  $\vec{q} = 0$  gives rise to a very small contribution to  $Q$ , which is proportional to  $V^\delta$ ,  $\delta > 1$ .<sup>11</sup>

However, if periodicity is taken into account, then the areas with  $\omega(\vec{q}) \rightarrow 0$  emerge near all points in the  $\vec{q}_\perp$  plane, which are equivalent to the origin. If one puts  $\vec{q}_\perp = \vec{g} + \tilde{q}_\perp$ , where  $\tilde{q}_\perp$  is the wave vector within the first Brillouin zone and  $\vec{g}$  is the wave vector of the reciprocal lattice, then these points are located at  $\tilde{q}_\perp \rightarrow 0$ . It is worth to mention that values of  $q_x V$  near these points are  $(g_x + \tilde{q}_x)V \approx 2\pi n V/a_v$ . The corresponding area in the  $\vec{q}_\perp$  plane is now given by  $\omega(\vec{q}) = 2\pi n V/a_v$ .

The value of  $g_x$  is limited by the cutoff  $g_x \leq 1/\xi$  or  $n \leq (a_v/2\pi\xi)$ . Hence, the condition of the area smallness in the  $\vec{q}_\perp$  plane with substitution of  $q_x \sim g_x \sim 1/a_v$  in  $q_x V$  and with  $\tilde{q}_\perp$  in  $C_1, C_2$  without changes takes the form:

$$a_v^2 c_\perp (c_{44} q_z^2 + c_\perp \tilde{q}_\perp^2) = (VH/\xi)^2. \quad (20)$$

Thus, these regions can be small being compared to the size of the first Brillouin zone at velocity  $V \ll V_c$ , where  $V_c = \xi \bar{c}/H \sim \omega_D \xi$ , where  $\bar{c}$  is some combination of  $c_{11}, c_{66}$  (more accurate expression for  $V_c$  will be given below, see Eq. (25)). A rough estimate of the characteristic velocity  $V_c$  can be done through the Debye frequency,  $V_c \sim \omega_D \cdot \xi$ . The Debye frequency appears because of periodicity of the dispersion law and the coherence length indicates that we take into consideration all  $\vec{q}_\perp$  wave vectors from  $1/a_v$  to  $1/\xi$ . (Note  $V_c$  is much smaller than the characteristic phase velocity of collective modes,  $V_{ph} \sim \omega_D \cdot a_v$ .)

The integral in (18) breaks into the sum over different  $\vec{g}$ . In the framework of a macroscopical approach the above derived formula (18) can be presented with coefficients  $C_1, C_2$  and  $\Delta$  that acquire macroscopic form:  $C_1/a_v^2 = c_{11} q_\perp^2 + c_{44} q_z^2$ ,  $C_2/a_v^2 = c_{66} q_\perp^2 + c_{44} q_z^2$  as were defined above and  $\Delta = (c_{11} - c_{66}) q_y^2 a_v^2$ ,  $C_1 - C_2 - \Delta = (c_{11} - c_{66}) q_x^2 a_v^2$ . Then, the last bracket in the numerator, after replacement  $\vec{q} \rightarrow \vec{g}$ , is equal to zero, that simplifies integral calculation. All  $\vec{q}$  vectors those are not included

in  $C_1, C_2$  or  $\Delta$  should be replaced by corresponding  $\vec{g}$  vectors then one can substitute in (18) both  $g_x = (2\pi/a_v) n$  instead of  $q_x$  and  $g_\perp = (2\pi/a_v) (n^2 + m^2)^{1/2}$  instead of  $q_\perp$ .

The sum over  $m$  and  $n$  may be replaced by the integral as we consider large  $m$  and  $n$ ,  $m, n \leq a_v/\xi$  and we integrate over  $\vec{q}$  only a small region near each vector of reciprocal lattice,  $|\vec{q}| \leq V/V_c$ , and considering large enough values of  $\vec{g}$  till  $1/\xi$ . The expression for the drag force per single vortex in the limit of small viscosity,  $H \gg \beta$ , and small velocities,  $V \ll V_c$ , finally is

$$F = \gamma_H V^{1/2}, \quad (21)$$

$$\gamma_H = \frac{2\pi\alpha^2 C_{imp}}{3\sqrt{c_{11}c_{44}c_{66}}} \frac{J\sqrt{H}}{a_v \xi^{3/2}} \cdot \begin{cases} 3/\sqrt{2}, & c_{11} \sim c_{66} \\ (c_{11}/c_{66})^{1/4}, & c_{11} \gg c_{66}, \end{cases}$$

where  $J = \int \xi^{3/2} dq_x dq_y f^2(g_\perp) g_x^3 g_\perp^2$  is the constant of the order of unity.

In the limit of high viscosity,  $\beta \gg H$ , and  $V \rightarrow 0$  the similar estimation demonstrates that again only small areas near  $\vec{g}$  are important and calculations yield

$$F = \gamma_\beta V^{1/2}, \gamma_\beta = \frac{\sqrt{2}\pi\alpha^2 C_{imp}}{\sqrt{c_{44}}} \frac{J\sqrt{\beta}}{a_v \xi^{3/2}} \left( \frac{1}{c_{66}} + \frac{1}{c_{11}} \right), \quad (22)$$

where  $J$  is the same constant as in (21).

So for small velocities,  $V \ll V_c$ , and any relation between  $H$  and  $\beta$  the drag force is  $F(V) = \gamma(H, \beta)\sqrt{V}$ , where the coefficient  $\gamma(H, \beta)$  in limiting cases is given by (21, 22).

When  $V \sim V_c$  one has to take into account not only small regions of the wave vectors near  $\vec{g}$ , but perform integration over all  $|\vec{q}| \leq 1/\xi$  as well. But this is an extremely complicated problem and we are not able to solve it even numerically as the detailed behaviour of  $C_{ik}$  at large  $\vec{q}$  is not known. Far above the critical velocity,  $V \gg V_c$ , one can significantly simplify the problem. In this case, it is possible to neglect contribution of the terms like  $(c_{11}, c_{66})q_\perp^2$  having the order of the value  $\omega_D$  to the denominator in (18), whereas the terms  $c_{44}q_z^2$ ,  $\beta q_x V$ ,  $H q_x V$  should be taken into consideration. Then, the calculations give at  $V > V_c$

$$F = \frac{\eta(H, \beta)}{\sqrt{V}}, \quad \eta = \frac{\pi\alpha^2 C_{imp} J' A(H, \beta)}{\sqrt{c_{44}a_v} \xi^{1/2}}, \quad (23)$$

where  $A(H, \beta) = 1/\sqrt{2H}$  at  $H \gg \beta$  and  $A(H, \beta) = 1/(2\sqrt{\beta})$  at  $H \ll \beta$ , the constant  $J' = \int \xi^{1/2} dq_x dq_y f^2(q_\perp) q_x^2 q_\perp^2 \sim J$ . Below the difference between  $J$  and  $J'$  will be disregarded. The dependence  $F \propto 1/\sqrt{V}$  corresponds to the single vortex limit,<sup>8</sup> if one would replace  $c_{44}a_v^2 \rightarrow \kappa$ . It is not surprising because in the limit  $V \gg V_c$  the only contribution the  $q_z$  wave vector to  $\omega(\vec{q})$  must be taken into account. This is exactly the case of the single vortex, for which  $q_x$  and  $q_y$  components

of the momentum  $\vec{q}$  are absorbed by the vortex, while elementary excitations propagating along the vortex axis acquire the momentum  $q_z$  and the energy  $q_x V$ , that does not restrict the integration area over  $\vec{q}$  (in contrast to (19) or (20)).

Thus, the velocity dependence of the additional drag force, which results from the vortex-defects interaction can be described as  $F = \gamma(H, \beta)\sqrt{V}$ ,  $V < V_c$ , and  $F = \eta(H, \beta)/\sqrt{V}$ ,  $V > V_c$ , where the coefficients  $\gamma$  and  $\eta$  are derived in Eqs. (21-23). To describe the total friction force the bare contribution  $\vec{F}_{bare} = -\beta\vec{V}$  should be concerned. The total friction force acting on the moving vortex lattice having the different velocity dependence for small and large velocities can be presented as  $\vec{F} = -(\vec{V}/V)F(V)$ , with the interpolating equation for  $F(V)$

$$F(V) = \frac{\gamma(H, \beta)\sqrt{V}}{1 + V/V_c} + \beta V, \quad (24)$$

where we use  $V_c = \eta(H, \beta)/\gamma(H, \beta)$  as a quantity characteristic of the problem. The schematic picture of the  $F(V)$  is presented in Fig.3.

This value does not contradict to estimates provided above in which only the order of  $V_c$  has been considered. From (21-23) one can extract the characteristic velocity  $V_c$  in limiting cases

$$V_c = \frac{\xi}{2\sqrt{2}} \cdot \begin{cases} (\sqrt{2}/H)\sqrt{c_{11}c_{66}}, & H \gg \beta, \quad c_{11} \sim c_{66}, \\ (3/H)(c_{11}c_{66}^3)^{1/4}, & H \gg \beta, \quad c_{11} \gg c_{66}, \\ c_{11}c_{66}/[\beta(c_{11} + c_{66})], & \beta \gg H. \end{cases} \quad (25)$$

In the problem one more characteristic velocity emerges,  $V_{trans}$ , at which the bare dissipation starts to be dominant compared to the additional one, i.e.  $\beta V \geq \gamma_H \sqrt{V}/(1 + V/V_c)$  at  $V > V_{trans}$ . We would like to emphasize that  $V_{trans}$  is determined as a velocity below that the additional drag force prevails over the bare drag force, whereas  $V_c$  is defined only by the additional drag force behaviour. In virtue of this,  $V_c$  and  $V_{trans}$  depend on the problem parameters in different manners. As it is seen from (25),  $V_c$  does not depend on the lattice-defects interaction intensity, i.e. does not comprise  $C_{imp}$  or  $\alpha$ . On the contrary,  $V_{trans}$  is directly correlated with the intensity  $\alpha$ . Because of the nonmonotonic velocity dependence of the additional drag force,  $V_{trans}$  exists for any as small as one likes value of  $C_{imp}\alpha^2$  and only the character of the total drag force,  $F(V)$ , is changed.

If the lattice-defects interaction intensity is small, while the bare constant  $\beta$  is large, then  $V_{trans}$  falls on the increasing part of  $F(V)$  and  $V_{trans} = (\gamma/\beta)^2$ . It corresponds to the case  $V_{trans} \ll V_c$  and the velocity  $V_c$  does not manifest itself at all:  $F(V) \propto \sqrt{V}$  at  $V \ll V_{trans}$  and  $F(V) \propto V$  at  $V > V_{trans}$ .

In the opposite case  $V_{trans} \gg V_c$ ,  $F(V) \propto \sqrt{V}$  at  $V < V_c$  and  $F(V) \propto 1/\sqrt{V}$  at  $V_c \ll V \ll V_{trans}$ ,

while the linear asymptotic  $F(V) \propto V$  appears only at  $V > V_{trans} \gg V_c$  and  $V_{trans} = (\eta/\beta)^{2/3}$ . Obviously that namely in the latter case the nonmonotonic  $F(V)$  dependence may be realised.

The concrete values of this velocity and the relation between  $V_c$  and  $V_{trans}$  will be estimated below.

### III. FORCED MOTION OF THE LATTICE

In the previous section we have employed a model of a rather weak random force induced by defects acting on the lattice. Now we proceed to description of the vortex lattice dynamics within a somewhat broader macroscopical approach. Having use the macroscopical form of the average friction force (24) instead of the random vortex-defects force considered in (6) we are able to reformulate the vortex lattice dynamics problem on the macroscopical basis.

Consider the displacement vector of vortices in the lattice  $\vec{U} = \vec{U}(\vec{r}, t)$  assuming that  $\vec{U}$  changes insignificantly over the scale of the lattice constant. We use the trivial averaging of the Lagrangian (1) with the dynamical part presented as in (6) and with substitutions  $\sum_l(\dots) \rightarrow \int dxdy/a_v^2(\dots)$  and  $\vec{u}_l(z, t) \rightarrow \vec{U}(\vec{r}, t)$ . The energy of lattice deformation is described by the elasticity theory (3). We assume the presence of the external force,  $F_e$ , while the effective drag force is determined by Eq.(24) with the change  $\vec{V} \rightarrow \dot{\vec{U}}$ .

The equation of motion takes the similar form as (6):

$$H(\vec{z} \times \dot{\vec{U}}) + \dot{\vec{U}} \frac{F(|\dot{\vec{U}}|)}{|\dot{\vec{U}}|} + \frac{\partial W\{\vec{U}\}}{\partial \vec{U}} = \vec{F}_e. \quad (26)$$

Using the equation (26) as the base we consider macroscopical dynamics of the vortex lattice, in particular, its stability against small perturbations. We present translational lattice velocity,  $\vec{U} = V$ , as a sum of the steady velocity of the lattice,  $\vec{V}_0$ , and the small correction  $\vec{u}$ ,  $\vec{U} = \vec{V}_0 + \vec{u}(\vec{r}, t)$ .

First we seek the solution for the lattice steady velocity  $\vec{V}_0$ . It allows us revealing  $F_e(V)$  dependence, which is merely current-voltage ( $I - V$ ) characteristic of the superconductor, see Ref.16. Because of the Hall term in (26) the steady velocity is convenient to express via components  $\vec{V}_0 = V_{\parallel}\vec{e}_{\parallel} + V_{\perp}\vec{e}_{\perp}$ , where  $\vec{e}_{\parallel} = \vec{F}_e/|\vec{F}_e|$ ,  $\vec{e}_{\perp} = (\hat{z} \times \vec{e}_{\parallel})$ . To describe the stationary flow of the vortex lattice with velocity  $\vec{V}_0$  a couple of equations of motion are found,

$$HV_{\perp} + \frac{F(V_0)V_{\parallel}}{V_0} = F_e, \quad -HV_{\parallel} + \frac{F(V_0)V_{\perp}}{V_0} = 0. \quad (27)$$

From these equations one can derive the components of stationary velocity  $V_{\perp}$  and  $V_{\parallel}$  via  $V_0$ :

$$V_{\perp} = \frac{F_e/H}{1 + (F/HV_0)^2}, \quad V_{\parallel} = \frac{FF_e/H^2V_0}{1 + (F/HV_0)^2}. \quad (28)$$

For simplicity we use  $F' = [dF(V)/dV]|_{V=V_0}$ ,  $F = F(V_0)$ . With the help of (28) the Hall angle could be given as

$$\tan \alpha = \frac{V_{\perp}}{V_{\parallel}} = \frac{HV_0}{F(V_0)}. \quad (29)$$

So if the drag force is a linear function of the translational velocity  $V_0$ ,  $F = \beta V_0$  (the usual case), then  $\tan \alpha$  does not depend on the velocity and the Hall angle tangent is merely the ratio of the Hall constant and the dissipation constant  $\beta$ . However, by considering the specific drag coefficient in Eq.(24) a striking feature appears, namely a velocity dependence of the Hall angle. In the limit  $V \rightarrow 0$   $\tan \alpha = \sqrt{V_0}/\gamma(\beta, H)$ , and the higher the lattice velocity the Hall angle larger, then it is saturated above the characteristic velocity  $V_{trans}$ .

By solving the Eq.(28) with the taking into account the drag force from Eq.(24) the dependence of  $F_e(V)$  acquires the following form:

$$V_0^2 H^2 + \left[ \frac{\gamma \sqrt{V_0}}{1 + V_0/V_c} + \beta V_0 \right]^2 = F_e^2 \quad (30)$$

The Eq.(30) has a simple form of the vector sum of two forces: the Hall force (the first term on the left hand side) and the drag force (the second term). The differential mobility related to the external force is

$$\frac{dV_0}{dF_e} = \frac{F_e}{V_0 H^2 + F(V_0)F'(V_0)}. \quad (31)$$

If one puts  $H = 0$  in (31) then the differential mobility  $dV_0/dF_e$  coincides with  $(dF/dV_0)^{-1}$ . But for  $H \neq 0$ , especially at  $H \gg F(V)/V$  these two quantities differ substantially.

The negative value of the differential mobility is usually considered as a condition (both necessary and sufficient) of the instability appearance. Nevertheless, it is quite possible that  $dV_0/dF < 0$ , whereas  $dV_0/dF_e > 0$ , see (31). Therefore, one has to reconsider the instability criterion for the case of macroscopic dynamics in the presence of gyroforce. This problem is of interest not only Abrikosov lattice dynamics, but it is also important for dynamics of any system having gyroforce and a non-monotonic velocity dependence of the drag force,  $F(V)$  (for instance, vortices in  $\text{He}^4$ , Bloch lines, etc.)

To investigate the stability of the vortex lattice we take  $\vec{U} = \vec{V}_0 t + \vec{u}(\vec{r}, t)$  and find the solution in the form  $\vec{u}(\vec{r}, t) \propto \vec{u}_q \exp(\Lambda t + i\vec{q}\vec{r})$ , where  $\Lambda$  defines the character of time evolution of small deviations from the steady motion. If the real part of  $\Lambda$  is negative, the value of  $-\text{Re}\Lambda > 0$  serves as a damping coefficient. Otherwise, if

the real part of  $\Lambda$  is positive for some values of  $\vec{q}$ , corresponding small deviations cause the motion instability with the instability increment  $\text{Re}\Lambda > 0$ .

For the small correction  $\vec{u}$  the couple of equations of motion are

$$\begin{aligned} u_{q,x} [\Lambda F'(V) + C_{xx}] + u_{q,y} (C_{xy} - \Lambda H) &= 0; \\ u_{q,x} (C_{xy} + \Lambda H) + u_{q,y} [\Lambda F(V)/V + C_{yy}] &= 0, \end{aligned} \quad (32)$$

where  $C_{ik} \equiv C_{ik}(\vec{q})$  are the coefficients used in (13), strictly speaking, their presentation in the longwave limit,  $V$  is the translational velocity of the lattice (for simplicity we omit here and further the index 0). It is more convenient to use in Eq.(32)  $\vec{e}_x$ ,  $\vec{e}_y$  instead of  $\vec{e}_{\parallel}$ ,  $\vec{e}_{\perp}$ . It is necessary to stress that this equation contains  $F(V)$  and  $F'(V) = dF(V)/dV$ , where  $F$  is the total drag force, rather than differential mobility  $dF_e/dV$ . As it will be demonstrated below, this is a very essential point which leads to the fact that the instability condition is not related directly to the usual one,  $dF_e/dV < 0$ .

The characteristic equation for  $\Lambda$  takes the form:

$$\begin{aligned} \Lambda^2 [F'(F/V) + H^2] + \omega^2(\vec{q})H^2 + \\ + \Lambda [F'C_{yy} + (F/V)C_{xx}] &= 0, \end{aligned} \quad (33)$$

where we used the equation  $C_{xx}C_{yy} - C_{xy}^2 = \omega^2(\vec{q})H^2$ ,  $\omega(\vec{q})$  is the dispersion law for free vortex oscillations (9) obtained above. The equation (33) possess two solutions:

$$\Lambda = -\Gamma(q) \pm i\Omega(q), \quad (34)$$

where

$$\begin{aligned} \Gamma(q) &= \frac{(F/V)C_{xx} + C_{yy}F'}{H^2 + (F/V)F'}, \\ \Omega^2(q) &= \frac{H^2\omega^2(q)}{H^2 + (F/V)F'} - \Gamma^2(q). \end{aligned} \quad (35)$$

In order to get the instability condition we analyse Eqs.(34, 35). If  $F' > 0$ , then  $\Gamma > 0$  and small oscillations decay for all values of  $\vec{q}$ . These are small damped oscillations with the frequency  $\omega \approx \Omega(\vec{q})$  at  $\Omega \ll \Gamma$ , otherwise, for  $\Omega \gg \Gamma$ , this is the case of overdamped dynamics of  $\vec{u}$ . Thus, the inequality  $F' < 0$  is the *necessary* condition for the lattice instability.

But if  $F' < 0$ , it is not *sufficient* to induce the lattice instability. The further analysis of Eqs.(34, 35) reveals two different and independent sufficient conditions imposed by the instability requirement  $\text{Re}\Lambda > 0$ .

The first condition is  $H^2 + (F/V)F' < 0$  or

$$-F'(F/V) > H^2. \quad (36)$$

This requirement is equivalent to the inequality  $dF_e/dV < 0$ , i.e. the rise of the negative differential mobility, in a literal sense. When this condition is satisfied the values of  $\Lambda$  in (34) are real and the instability

appears at any  $\vec{q} \neq 0$ . [If one considers the inertial term in the equation of motion (26), then even the uniform deviations  $U(t)$  from  $U_0$  (dynamics with  $\vec{q} = 0$ ) become unstable when  $dF_e/dV < 0$ .]

Hence, the fulfillment of the condition  $dF_e/dV < 0$  apparently indicates the rise of the instability of the lattice translational motion. But this condition is substantially more strict than  $F'(V) < 0$  (see Fig. 3), and as we will demonstrate in the next Section never will be satisfied in our model (this condition, though, could be met in other models with  $dF/dV < 0$ ).

The second condition corresponds to the inequality:

$$-F'C_{yy} > (F/V)C_{xx}. \quad (37)$$

The condition (37) is one more sufficient condition of the instability independent of (36). Depending on the strength of requirements (36,37) one of them should be fulfilled earlier than the other. We will demonstrate below that the condition (37) will be met in real superconductors unlike the requirement (36).

#### IV. DISCUSSION

In the previous section we have established a set of conditions of the vortex lattice instability which comprises both the necessary and sufficient requirements. Let us estimate the least strict condition, the necessary condition  $F'(V) < 0$ . By differentiating (24) one obtains

$$\frac{\gamma}{2\sqrt{V}} \frac{(V/V_c - 1)}{(1 + V/V_c)^2} > \beta. \quad (38)$$

The left hand side of the inequality (38) as a function of the variable  $V/V_c$  has the maximal value  $0.04\gamma/\sqrt{V_c}$  at  $V/V_c \simeq 2$ . Using  $V_c = \eta/\gamma$  one can rewrite (38) as

$$\beta \leq 0.04 \cdot \gamma^{3/2}/\sqrt{\eta}. \quad (39)$$

The parameters  $\eta$  and  $\gamma$  can be taken from the limiting formulae (21-23) and expressed through microscopic characteristics of the superconductor (see, e.g., Refs. 1, 16): the elastic moduli  $c_{11} = c_{44} = B^2/4\pi$ ,  $c_{66} = B\Phi_0/(8\pi\lambda)^2$ , where  $B$  is the magnetic induction,  $\Phi_0$  is the flux quanta,  $\lambda$  is the penetration depth. The vortex lattice constant is determined by magnetic induction  $a_v^2 = (2/\sqrt{3})(\Phi_0/B)$ . The Hall constant is  $H = \Phi_0 en/c$ , where  $e$  is the electron charge,  $c$  is the speed of light,  $n$  is the density of superconducting electrons,  $n = mc^2/4\pi\lambda^2 e^2$  with the electron mass,  $m$ . The parameter  $\alpha$  can be written as  $\alpha = a^3 (H_{cm}^2/8\pi) (\Delta T_c/T_c)$ , where  $H_{cm}$  is the superconducting thermodynamic critical field,  $\Delta T_c/T_c$  is a relative suppression of the critical temperature nearly a defect,  $a^3$  is the volume of the defect ( $a$  is of the order of the interatomic distance). As a result, in the case of the small dissipation, the Eq.(39)

takes the following form:

$$\frac{\beta}{H} \lesssim 10^{-2} (C_{imp} a^3) \cdot \left( \frac{\Delta T_c}{T_c} \right)^2 \frac{a^3 \lambda^3}{\xi^6} \left( \frac{H_{c1}}{B} \right)^p, \quad (40)$$

where  $H_{c1} = \Phi_0/4\pi\lambda^2$  is the quantity of the order of the lower critical field and  $p$  depends on  $(c_{11}/c_{66})$

$$p = \begin{cases} 19/8, & c_{11} \gg c_{66}, \\ 11/4, & c_{11} \sim c_{66}. \end{cases} \quad (41)$$

If one would take the numerical values of  $\lambda$  and  $\xi$ , as for typical high  $T_c$  superconductor YBCO:<sup>1</sup>  $\lambda = 10^{-5}$  cm,  $\xi = 10^{-7}$  cm,  $a = 5 \cdot 10^{-8}$  cm,  $\Delta T_c/T_c \approx 1$ ,  $C_{imp} a^3 \approx 10^{-3}$  then

$$(H_{c1}/B)^p \geq \frac{\beta}{H}. \quad (42)$$

This condition is met at magnetic fields close to the lower magnetic field,  $H_{c1}$ , and the microscopical parameters used above for YBCO compound. The maximal values of  $\beta = \Phi_0^2/2\pi\xi^2 c^2 \rho_n$ , where  $\rho_n$  is the normal conductivity of the superconductor. So the maximal value of the ratio  $\beta/H$  is  $\beta/H = (1/\rho_n)(\lambda/\xi)^2 (\hbar/2\pi mc^2)$  and at  $\rho_n = 5 \cdot 10^{-16}$  s (the typical value for YBCO) it is equal approximately to  $10^{-2}$ . Note this is just the upper limit of  $\beta/H$  and for real superconductors this value can be even smaller. Thus, for the case of low viscosity the instability necessary condition may be fulfilled at high enough values of magnetic fields.

In the case of high viscosity,  $H \ll \beta$ , the necessary condition can be satisfied with difficulty as  $\gamma(\beta)$  contains  $\beta$  and the right hand side of the inequality (38) becomes unity (in contrast to the case  $H \gg \beta$  where the small parameter  $\beta/H$  takes place).

Now we study the first sufficient condition  $-F'(F/V) > H^2$ . We carry out the calculation in the similar manner as for (39) and obtain (neglecting the bare dissipation as  $H \gg \beta$ ) that  $0.1 \cdot \gamma^{3/2}/\sqrt{\eta} > H$ . Apparently this condition cannot be satisfied since though the same inequality as (42) is valid for this condition, but on the right hand side instead the small parameter  $\beta/H$  unity is found. Then, we consider the second sufficient condition  $-F'/(F/V) > C_{xx}/C_{yy}$  which acquires the following form:

$$\frac{\gamma}{2\sqrt{V}} \frac{((C_{yy} - 2C_{xx})V/V_c - (C_{yy} + 2C_{xx}))}{(C_{yy} + C_{xx})(1 + V/V_c)^2} > \beta. \quad (43)$$

It is seen that the inequality (43) has an obvious resemblance with Eq.(38). The difference is caused by  $\vec{q}$ -dependent coefficients containing  $C_{xx}$  and  $C_{yy}$  in Eq.(43).

For  $2C_{xx} > C_{yy}$ , which is realised, for instance, when  $c_{11} = c_{66}$ , the left hand side of (43) is negative. Therefore, for some values of  $C_{xx}$ ,  $C_{yy}$  the condition (43) cannot be fulfilled and no instability appears. However, by using the fact that  $c_{66} \simeq (c_{11}/4)(H_{c1}/B) \ll c_{11}$  and choosing an appropriate  $\vec{q}$  it can be satisfied. The simple



analysis shows that when  $c_{11} \geq c_{66}$  the most favorable case is the wave vectors along the  $y$ -axis,  $\vec{q} \parallel \vec{e}_y$ . Then, the inequality (43) takes the following form:

$$\frac{\gamma}{2\sqrt{V}} \frac{((c_{11} - 2c_{66})V/V_c - (c_{11} + 2c_{66}))}{(c_{11} + c_{66})(1 + V/V_c)^2} > \beta. \quad (44)$$

The maximal value of left hand side of (44) is attained at  $3(c_{11} - 2c_{66})(V/V_c) = (3c_{11} + 4c_{66} + 2\bar{c})$ , where  $\bar{c} = \sqrt{3c_{11}^2 + 6c_{11}c_{66} + c_{66}^2}$ . It is equal to

$$\frac{\gamma^{3/2}}{\sqrt{\eta}} \frac{(3\sqrt{3}/4)(c_{11} - 2c_{66})^{5/2}(\bar{c} - c_{66})}{(c_{11} + c_{66})\sqrt{3c_{11} + 4c_{66} + 2\bar{c}}(3c_{11} - c_{66} + \bar{c})^2}. \quad (45)$$

Thus, the sufficient condition (44) is more strict than necessary one given by (38) owing to the multipliers with  $c_{11}$  and  $c_{66}$ . In particular, it can be met only at  $c_{11} > 2c_{66}$ . However, in the case of interest  $c_{11} \geq 10 c_{66}$ , the necessary and sufficient conditions are close each other and the sufficient requirement (44) can be satisfied as well. As the instability is expected to arise when  $H \gg \beta$  and  $V \approx 2V_c$ , one can estimate  $V_c$  for this case. In accordance with Eq.(25) and with the use of the above presented microscopical parameters for YBCO  $V_c$  can be presented as:

$$V_c \approx 10^{-2} \cdot (B[\text{Oe}])^{3/2}, \quad (46)$$

and at  $B = 10^3$  Oe,  $V_c \approx 300$  cm/s. Thus, the instability may appear under the rather strict condition (42) and at the velocities of the order of  $2V_c \sim 10^3$  sm/s. Hence, both the necessary and sufficient conditions can be fulfilled at low viscosity,  $\beta \ll H$ , relatively small magnetic fields and quite high velocities of the vortex lattice motion.

At small velocities another effect, namely a nonlinear  $I$ - $V$  characteristic of the superconductor emerges. This effect does not impose limits for superconductor parameters. It can be observed at  $\beta > H$  and at strong magnetic fields, when  $V_c \gg V_{trans}$  and no instability arises.

At  $V_c \gg V_{trans}$ ,  $F(V)$  dependence is monotonic and contains only part with  $F \propto \sqrt{V}$ ,  $V < V_{trans}$ ,  $F \propto V$ ,  $V > V_{trans}$ . In this case, the quadratic  $I$ - $V$  characteristic is realised at  $V < V_{trans}$ . This effect can be found at very small defect concentration and rather dense lattice. For instance, if  $C_{imp}a^3 = 10^{-6}$ ,  $V_{trans} \approx 10$  cm/s when  $H \gg \beta$  and  $V_{trans} \approx 70$  cm/s for  $H \ll \beta$  at  $B = 10^3$  Oe.

## V. CONCLUSION

We have demonstrated that due to the interaction of weak defects with the Abrikosov vortex lattice the additional channel of dissipation appears, which contributes to the total dissipation of the moving lattice. As a result the additional friction force has a non-linear dependence of the translational lattice velocity,  $F \propto \sqrt{V}$  for both weak bare dissipation,  $H \gg \beta$ , and high viscosity  $H \ll \beta$ . This channel of dissipation turns out to be an essential source of the system dissipation and even prevails over the bare dissipation,  $\beta V$ , below the characteristic velocity  $V_{trans}$ .

The nonlinear character of the additional drag force results in a nonlinear current-voltage characteristic of the superconductor. Besides, the tangent of the Hall angle  $\tan \alpha$  appears to be the translational lattice velocity  $V_0$  dependent up to the characteristic velocity  $V_{trans,H}$ , above which it is saturated.

Studying the stability of the vortex lattice against uniform and nonuniform perturbations we have found several salient features.

The instability of the vortex lattice motion is found to occur in the limit of small viscosity,  $H \gg \beta$ . We have revealed that the negative differential mobility,  $dV_0/dF_e$ , is not *a priori* the lattice motion instability requirement. Due to the Hall component in the lattice motion, the value  $[dF(V)/dV]^{-1}$  does not coincide with the differential mobility  $dV/dF_e$ . The estimates suggest that condition  $dV/dF_e < 0$  can be met with difficulty in contrast to  $dV/dF < 0$ . The instability of the lattice motion regarding nonuniform perturbations has been found when the value of  $(-dF/dV)$  is greater than the combination of the elastic moduli  $c_{11}$  and  $c_{66}$ , while the more strict requirement  $dV/dF_e < 0$  is still not fulfilled. It is worth to note that the vortex lattice instability concerned in this paper has different origin than instabilities predicted in Ref. 17.

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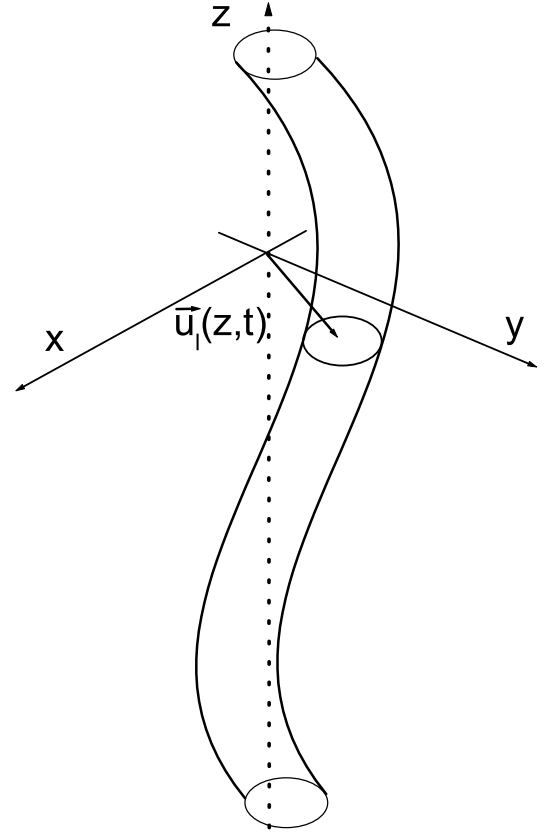


FIG. 1: The sketch of the vortex displacement  $\vec{u}_l(z, t)$ .

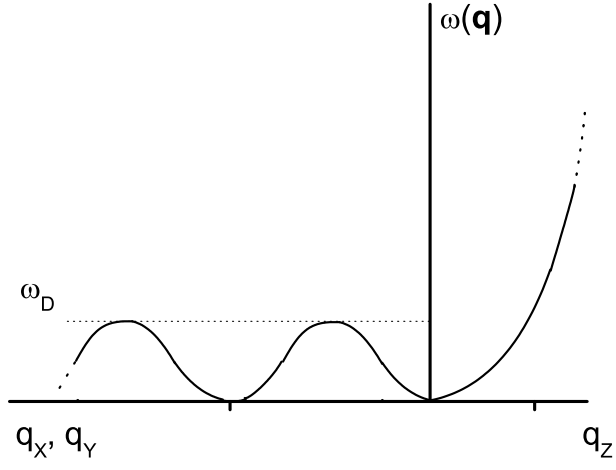


FIG. 2: The schematic representation of the dispersion law of the vortex lattice. The dotted line indicates the Debye frequency,  $\omega_D$ .

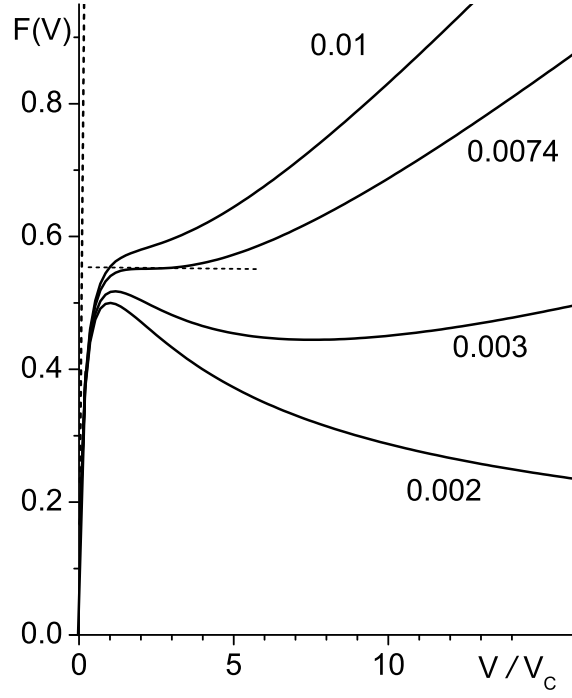


FIG. 3: The velocity dependencies of the drag force  $F(V)$  at  $B = 3H_{c1}$  for different values of the ratio  $\beta/H$  (its value is shown near each curve). The dotted line parallel to the  $x$ -axis marks the point on the curve with the critical ratio  $\beta/H = 0.0074$  where the nonmonotonic character of  $F(V)$  starts to develop. The dashed line describes the relation of the external force,  $F_e$  and the velocity  $V$ , at very small value of the ratio  $\beta/H = 0.002$ . As was noted in the text the behaviour of this curve is very far from nonmonotonic and can be hardly distinguished from the  $y$ -axis.